

1. A mass $3m$ moves on a frictionless horizontal surface with speed v_0 toward a stationary mass $2m$. To the right of the mass $2m$ is an ideal spring of spring constant k , whose other end is fixed to a wall.

[12 Pts.]

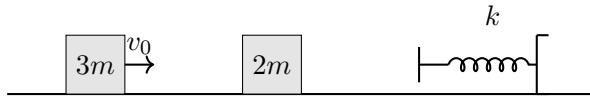
- (a) The collision between the two masses is *completely inelastic*. The combined object then continues to move and compresses the spring. Determine the maximum compression of the spring.
- (b) Now consider a different scenario: all collisions between the two masses are *perfectly elastic*, and the spring is placed far enough to the right so that the mass $3m$ never reaches it.

[7 Pts.]

- i. how many collisions occur between the two masses,

[6 Pts.]

- ii. the final velocities of both masses after all collisions have ceased.



a)

$$3mv_0 = (3m + 2m)v \Rightarrow v = \frac{3}{5}v_0$$

$$\frac{1}{2}5m \left(\frac{3}{5}v_0 \right)^2 = \frac{1}{2}kx^2 \Rightarrow x = \sqrt{\frac{9m}{5k}} v_0$$

b) **1st collision:**

$$3mv_0 = 3mv_1 + 2mv_2 \Rightarrow v_2 = \frac{3}{2}(v_0 - v_1) \quad \text{and} \quad \frac{1}{2}3mv_0^2 = \frac{1}{2}3mv_1^2 + \frac{1}{2}2mv_2^2$$

$$\frac{5}{2}v_1 = \frac{1}{2}v_0 \Rightarrow v_1 = \frac{1}{5}v_0 \quad \text{and} \quad v_2 = \frac{3}{2}\left(v_0 - \frac{1}{5}v_0\right) = \frac{6}{5}v_0$$

2nd collision: $2m$ moves to the left, $3mv'_1 - 2mv'_2 = \frac{12}{5}mv_0 - \frac{3}{5}mv_0 \Rightarrow v'_2 = \frac{3}{2}v'_1 - \frac{9}{10}v_0$

Energy is conserved $E_i = \frac{1}{2}3mv_0^2 = \frac{3}{2}mv_0^2$ then, $\frac{1}{2}3mv_1'^2 + \frac{1}{2}2mv_2'^2 = \frac{3}{2}mv_0^2$

$$v_2'^2 = \frac{3}{2}(v_0^2 - v_1'^2)$$

$$\left(\frac{3}{2}v'_1 - \frac{9}{10}v_0 \right)^2 = \frac{3}{2}(v_0^2 - v_1'^2)$$

$$v_1'^2 - \frac{18}{15}v_0v'_1 - \frac{23}{5 \times 25}v_0^2 = 0 \Rightarrow (v'_1 + \frac{1}{5}v_0)(v'_1 - \frac{23}{25}v_0) = 0$$

$$\Rightarrow \boxed{v'_1 = \frac{23}{25}v_0} \quad \text{and} \quad \boxed{v'_2 = \frac{12}{25}v_0}$$

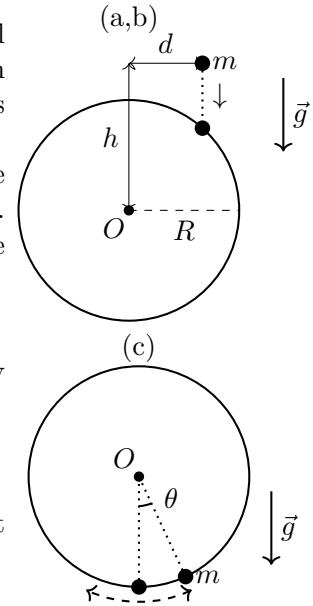
$2m$ reflected again from the spring but never catch up the other $\frac{12}{25}v_0 < \frac{23}{25}v_0 \Rightarrow 2 \text{ collisions}$

$$\boxed{v_{1f} = \frac{23}{25}v_0}$$

$$\boxed{v_{2f} = \frac{12}{25}v_0}$$

2. A solid disc of radius R and mass M is mounted on a vertical wall and can rotate without friction about a fixed horizontal axle through its center. The disc has moment of inertia $I_{\text{disc}} = \frac{1}{2}MR^2$ about this axis.

A small mass m is released from rest at a point a horizontal distance d to the right of the disc's center and a vertical distance h above it. The mass falls straight down under gravity g , strikes the rim of the disc, and sticks to it.



- [9 Pts.] (a) Find the angular velocity of the disc-mass system immediately after the collision.
- [9 Pts.] (b) Determine the mechanical energy lost in the collision.
- [7 Pts.] (c) For small oscillations of the disc-mass system about the lowest point, find the period of oscillation.

a)

$$mgh - mg\sqrt{R^2 - d^2} = \frac{1}{2}mv^2$$

$$\Rightarrow v = \sqrt{2g(h - \sqrt{R^2 - d^2})}$$

angular momentum: $mv d = I\omega = (I_d + mR^2)\omega = \left(\frac{1}{2}M + m\right) R^2\omega$

$$\omega = \frac{mdv}{\left(\frac{1}{2}M + m\right) R^2} = \frac{md\sqrt{2g(h - \sqrt{R^2 - d^2})}}{\left(\frac{1}{2}M + m\right) R^2}$$

b)

$$\frac{1}{2}mv^2 - \frac{1}{2}I\omega^2 - \frac{1}{2}mR^2\omega^2 = \frac{1}{2}mv^2 - \frac{1}{2}(I + mR^2)\omega^2$$

$$= mg(h - \sqrt{R^2 - d^2}) \left[1 - \frac{md^2}{\left(\frac{1}{2}M + m\right) R^2} \right]$$

c)

$$\tau = -mgR \sin \theta \approx -mgR\theta$$

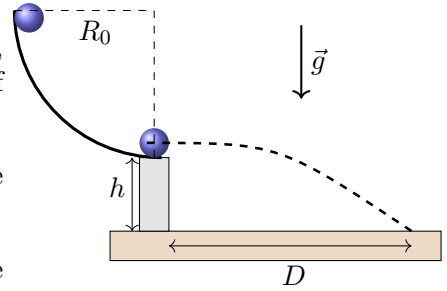
$$I \frac{d^2\theta}{dt^2} = -mgR\theta$$

$$\omega_{\text{osc}} = \sqrt{\frac{mgR}{I}} = \sqrt{\frac{mg}{(M/2 + m)R}}$$

$$T = \frac{2\pi}{\omega_{\text{osc}}} = 2\pi \sqrt{\left(1 + \frac{M}{2m}\right) \frac{R}{g}}$$

3. A ball of radius r and mass m rolls on the inside of a track of radius R_0 as shown in the figure. The gravitational acceleration is downward with magnitude g . ($I = \frac{2}{5}mr^2$)

- [12 Pts.] (a) If the ball starts from rest at the vertical edge of the track, what will be its speed when it reaches the lowest point of the track, rolling without slipping the whole way?
- [7 Pts.] (b) When it leaves the track, at what distance D from the base of the track will the sphere hit the ground?
- [6 Pts.] (c) What will be its angular speed when the sphere hits the ground?
(Ignore thickness of track.)



a)

$$mgR_0 = mgr + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2, \quad \omega = \frac{v}{r}$$

$$mg(R_0 - r) = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}mr^2\frac{v^2}{r^2}$$

$$g(R_0 - r) = \frac{7}{10}v^2$$

$$\Rightarrow v = \sqrt{\frac{10}{7}g(R_0 - r)}$$

b)

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}}$$

$$D = vt = \sqrt{\frac{20}{7}h(R_0 - r)}$$

c)

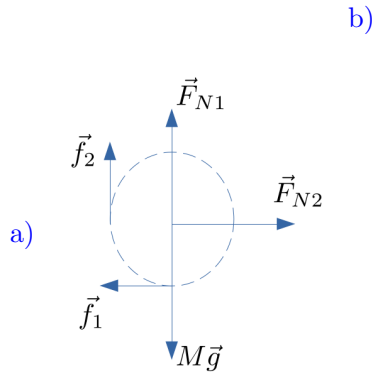
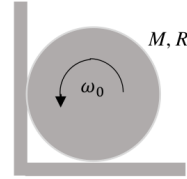
$$\omega = \frac{v}{r} = \sqrt{\frac{10}{7}g\frac{(R_0 - r)}{r^2}} \quad \text{no change in air}$$

4. A cylindrical disk of mass M and radius R ($I_{CM} = MR^2/2$) skids against both the horizontal and vertical surfaces of a corner as shown in the figure. The coefficient of kinetic friction between the disk and both surfaces is μ_k . The initial angular speed of the disk at the instant it is placed in the corner is ω_0 , and its axis of rotation does not move while it rotates.

[5 Pts.] (a) Draw a free body diagram for the disk.

[6 Pts.] (b) Find the net torque.

[14 Pts.] (c) Find the total number of revolutions the disk makes before stopping.



b)

$$a = 0 \quad \& \quad \alpha \neq 0 \quad \Rightarrow \quad \Sigma \vec{F} = 0 \quad \& \quad \Sigma \vec{\tau} \neq 0$$

$$F_{N2} - f_1 = 0 \quad \& \quad f_1 = \mu_k F_{N1}$$

$$F_{N1} + f_2 - Mg = 0 \quad \& \quad f_2 = \mu_k F_{N2}$$

$$-\mu_k F_{N1} + F_{N2} = 0$$

$$F_{N1} + \mu_k F_{N2} = Mg \quad \Rightarrow \quad F_{N1} = \frac{Mg}{1 + \mu_k^2} \quad \& \quad F_{N2} = \frac{\mu_k Mg}{1 + \mu_k^2}$$

$$\Rightarrow f_1 = \frac{\mu_k Mg}{1 + \mu_k^2}, \quad f_2 = \frac{\mu_k^2 Mg}{1 + \mu_k^2}$$

$$\Sigma \tau = Rf_1 + Rf_2 = \frac{RMg\mu_k(1 + \mu_k)}{1 + \mu_k^2} \quad \text{clockwise}$$

c)

$$\Sigma \tau = I\alpha \quad \Rightarrow \quad \alpha = \frac{\Sigma \tau}{I} = \frac{2g\mu_k(1 + \mu_k)}{R(1 + \mu_k^2)}$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta, \quad \omega = 0 \text{ when it stops}$$

$$\theta = \frac{\omega_0^2 R(1 + \mu_k^2)}{4g\mu_k(1 + \mu_k)}$$

$$\# \text{ of revolutions} = \frac{\theta}{2\pi} = \frac{\omega_0^2 R}{8\pi g} \left(\frac{1 + \mu_k^2}{\mu_k + \mu_k^2} \right)$$